Notes on Kitaev's Periodic Table for Topological Insulators and Superconductors

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1 Introduction

This note aims to give a mathematically worded version of Kitaev's [11]. That paper gives a systematic classification of topological insulators and superconductors, where these systems are understood to be so-called gapped free fermion systems and the classification is in terms of the symmetry and spatial dimension of the system. The symmetries under consideration are time-reversal and a symmetry concerned with particle number conservation.

This text started out of curiosity about table I in [8]. This table is called a "periodic table for topological insulators and superconductors" and the entries of part of it read $\mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}$ both along the rows (in reverse) and the columns. This of course sparks any topologist's interest.

The classification is carried out by showing that associated to such a system there is an element of a K-group, described in terms of bundles of modules for Clifford algebras. This description, due to Karoubi [9] seems inspired by the paper by Atiyah, Bott and Shapiro [1] from the sixties. That classical paper links a classification of Clifford algebras and periodicity of their representations with the Bott periodicity [4] for the stable homotopy of the classical groups.

In chapter 2 we recall the results from K-theory needed. Kiteav's text being a paper in condensed matter physics, the language used is that of the Hamiltonian formulation of quantum (statistical) field theory, chapter 3 of this note is devoted to giving an incomplete introduction to this. After this we give a mathematical description of gapped free fermion systems in chapter 4. We are then in a position to formulate the classification problem at hand and solve it using the results in from K-theory in the last part 5.

2 K-theory, Bott periodicity & Clifford algebras

We will briefly review the basics we need about the classical subject of topological K-theory and its connection with Clifford algebras. As was pointed out by Atiyah, Bott and Shapiro in their '64 paper [1], there is a striking connection between Clifford algebras and K-theory. They explained this connection by given an isomorphism between modules for Clifford algebras and the K-theory of a point. For the more general case of vector bundles with Clifford algebra actions, they give relation in terms of the Thom isomorphism. Later Karoubi [9] showed there was a more direct connection.

In this chapter, we will first review some basics on Clifford algebras, then recall the Atiyah-Hirzebruch definition for topological K-theory and finally describe Karoubi's version of K-theory and compare it to Atiyah's version.

2.1 Clifford Algebras

In this section we will treat definitions and some properties of Clifford algebras. The exposition roughly follows [1].

2.1.1 Definition and basic properties

The general definition of a Clifford algebra is easy to state:

Definition 1. Let k be a field. The *Clifford algebra* associated to a quadratic form Q on a k-module E is a quotient algebra of the tensor algebra $TE = \bigoplus_{n=0}^{\infty} E^{\otimes n}$, given by

$$C(Q) = TE/I(Q), \tag{1}$$

where $I(Q)$ is the ideal generated by the elements $x \otimes x - Q(x) \cdot 1$.

We will only concern ourselves with fields of characteristic zero.

Clifford algebras satisfy the following universal property, which we state without proof:

Proposition 2. *Let $f : E \to A$ be a map of k-algebras, where A has a unit, such that for all $x \in E$, we have that $f(x)^2 = Q(x)1_A$. Then there exists a unique extension $\tilde{f} : C(Q) \to A$ that makes the diagram*

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d, "i_Q"'] & A \\ C(Q) \arrow[ur, dashed, "\tilde{f}"] & \end{tikzcd} \tag{2}$$

commute. Here i_Q denotes the (injective) inclusion of E into $C(Q)$.

Clifford algebras come equipped with a $\mathbb{Z}_2$-grading, induced from the $\mathbb{N}$-grading on TE, the parity of the grading on TE is preserved under the quotient.

It will be useful to view Clifford algebras as finitely presented k-algebras.

Lemma 3. *Pick a basis $\{e_i\}$ for E, and let $Q_{ij} = Q(e_i, e_j)$. Then we can view $C(Q)$ as the $\mathbb{Z}_2$-graded algebra with unit generated by the odd generators e_i, subject to the relations that*

$$\{e_i, e_j\} := e_i e_j + e_j e_i = 2Q_{ij} \tag{3}$$

Proof. It is clear that the e_i generate TE. Consider the equalities

$$(e_i \pm e_j) \otimes (e_i \pm e_j) = Q_{ii} \pm 2Q_{ij} + Q_{jj} \tag{4}$$

in $C(Q)$. Subtracting these from each other yields (3), so the e_i indeed satisfy these relations. Conversely, if $v \in E$, we can write $v = \sum v_i e_i$ and

$$Q(v,v) = \sum_{i,j} Q_{ij} v_i v_j, \tag{5}$$

in the quotient this should be the same as

$$\sum_{i,j} (e_i e_j) v_i v_j = \sum_{i<j} (e_i e_j + e_j e_i) v_i v_j + \sum_i e_i e_i v_i^2 \tag{6}$$

$$= \sum_{i<j} 2Q_{ij} v_i v_j + \sum_i Q_{ii} v_i^2 = \sum_{i,j} Q_{ij} v_i v_j. \tag{7}$$

So we see the ideal generated by the relations in the generator picture is mapped isomorphically to the ideal $I(Q)$ of TE. $\square$

From this point of view, the following proposition is easy to prove:

Proposition 4. *Let $E = E_1 \oplus E_2$ be a decomposition of E such that $Q(e_1, e_2) = 0$ for all $e_1 \in E_1$ and $e_2 \in E_2$. Let $Q_i = Q|_{E_i}$, for $i = 1, 2$. Then we have*

$$C(Q) \cong C(Q_1) \otimes_{\mathbb{R}} C(Q_2), \tag{8}$$

where $\otimes$ denotes the graded tensor product, as appropriate for graded algebras.

Proof. Let $\{e_i\}_{i\in\{1,2,\ldots,p+q\}}$ be a basis for E such that the first p elements span E_1 and the last q span E_2. By the above description of Clifford algebras, we see that these basis elements generate $C(Q_1)$ and $C(Q_2)$, respectively, as well as $C(Q)$. The graded tensor product is generated by $\{e_i \otimes 1\}_{i\in\{1,2,\ldots,p\}}$ together with $\{1 \otimes e_i\}_{i\in\{p+1,p+2,\ldots,p+q\}}$. There is an obvious bijection on generators. It remains to show this bijection preserves the relations. Note that the anti-commutation relations amongst the first p basis elements are unchanged by tensoring with 1 from the right, and a similar argument holds for the last q. We are left with considering, for $i = 1, 2, \ldots, p$ and $j = p+1, p+2, \ldots p+q$:

$$\begin{aligned} \{e_i \otimes 1, 1 \otimes e_j\} &= e_i \otimes 1) \cdot (1 \otimes e_j) + (1 \otimes e_j) \cdot (e_i \otimes 1) \\ &= (-1)^{|1||1|} e_i \otimes e_j + (-1)^{|e_j||e_i|} e_i \otimes e_j \\ &= 0 = \{e_i, e_j\}. \end{aligned} \tag{9}$$

Here $|v|$ denotes the degree of the element v. This shows that the obvious bijections are maps of algebras in both directions, hence isomorphisms. $\square$

From here onward we restrict our attention to particular sequence of Clifford algebras $\mathrm{Cl}_{p,q} = C(Q_{p,q})$, with $p, q \in \mathbb{N}$, over the reals, with underlying vector space $\mathbb{R}^{p+q}$ and quadratic form $Q_{p,q}(x_1, x_2, \ldots, x_{p+q}) = \sum_{i=1}^{p} x_i^2 - \sum_{i=p+1}^{p+q} x_i^2$. By the above proposition, we have

$$\mathrm{Cl}_{p,q} \cong \mathrm{Cl}_{1,0} \otimes_{\mathbb{R}} \mathrm{Cl}_{1,0} \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} \mathrm{Cl}_{1,0} \otimes_{\mathbb{R}} \mathrm{Cl}_{0,1} \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} \mathrm{Cl}_{0,1}, \tag{10}$$

with p factors of $\mathrm{Cl}_{1,0}$ and q factors of $\mathrm{Cl}_{0,1}$. In particular, we have:

$$\mathrm{Cl}_{0,k+4} \cong \mathrm{Cl}_{0,k} \otimes_{\mathbb{R}} \mathrm{Cl}_{0,4}, \tag{11}$$

$$\mathrm{Cl}_{0,k+8} \cong \mathrm{Cl}_{0,k} \otimes_{\mathbb{R}} C_{0,8}. \tag{12}$$

The factors in (10) have a nice description, $\mathrm{Cl}_{0,1}$ is generated over $\mathbb{R}$ by 1 and a single odd element that squares to -1, so is isomorphic to the complex numbers. The algebra $\mathrm{Cl}_{1,0}$ is generated by 1 and e, with $e^2 = 1$, so is isomorphic to $\mathbb{R} \oplus \mathbb{R}$ with component-wise multiplication, sending $1 \mapsto (1,1)$ and $e \mapsto (1,-1)$.

2.1.2 The algebras $\mathrm{Cl}_{p,q}$

Let us determine the algebras appearing in the sequences $C_q = \mathrm{Cl}_{0,q}$ and $C'_p = \mathrm{Cl}_{p,0}$. For notational convenience, we will suppress the hats on the graded tensor product and assume it is clear from context what is meant. The key lemma here is:

Lemma 5. *We have isomorphisms:*

$$C_q \otimes_{\mathbb{R}} C'_2 \cong C'_{q+2} \tag{13}$$

$$C'_p \otimes_{\mathbb{R}} C_2 \cong C_{p+2} \tag{14}$$

Proof. For the first isomorphism, apply the universal property 2 to the map

$$f : \mathbb{R}^{q+2} \to C_q \otimes C'_2 \tag{15}$$

$$e_i \mapsto \begin{cases} e_{i-2} \otimes e'_1 e'_2 & \text{for } 2 < i \leq q \\ 1 \otimes e'_i & \text{for } 1 \leq i \leq 2 \end{cases}, \tag{16}$$

with obvious notation for the generators. The extension of this map to C'_{q+2} induces a bijection on generators and is a map of algebras by construction, so witnesses the sought-after isomorphism. The proof for the second isomorphism is analogous. □

Corollary 6. *We have the isomorphism:*

$$C_4 \cong C'_4, \tag{17}$$

With this lemma in hand, we can determine the algebras C_q and C'_p step by step. We already know $C_1 \cong \mathbb{C}$ and $C'_1 \cong \mathbb{R} \oplus \mathbb{R}$.

The algebra C_2 is generated by $\{1, e_1, e_2\}$, with $e_1^2 = e_2^2 = -1$ and $e_1 e_2 = -e_2 e_1$, so spanned as a vector space by $\{1, e_1, e_2, e_1 e_2\}$ and the multiplication on this vectors space is just the quaternionic multiplication. So $C_2 \cong \mathbb{H}$.

On the other hand, C'_2 is generated by a unit and two anticommuting elements that square to that unit. An explicit model for these generators are the two by two matrices over $\mathbb{R}$ with the identity as unit and the matrices

$$e'_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e'_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{18}$$

that anti-commute and multiply to

$$e'_1 e'_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{19}$$

These four matrices span the two by two matrices over $\mathbb{R}$, denoted by $\mathbb{R}(2)$. So $C'_2 \cong \mathbb{R}(2)$.

In iteratively applying our key lemma, we will need to compute tensor products. The following lemma lists the identities we need:

Lemma 7. *Let F be any of $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ and denote by $F(n)$ the algebra of $n \times n$ matrices over F. Then we have the following:*

$$\begin{aligned} F(n) &\cong \mathbb{R}(n) \otimes_{\mathbb{R}} F, \\ \mathbb{R}(n) \otimes_{\mathbb{R}} \mathbb{R}(m) &\cong \mathbb{R}(nm) \end{aligned} \tag{20}$$

$$\begin{aligned} \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} &\cong \mathbb{C} \oplus \mathbb{C} \\ \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} &\cong \mathbb{C}(2) \\ \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H} &\cong \mathbb{R}(4). \end{aligned} \tag{21}$$

The proof of this lemma is not very interesting.

First applying the key lemma with $k = 1, 2$, we see that

$$C_3 \cong (\mathbb{R} \oplus \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{H} \oplus \mathbb{H} \tag{22}$$

$$C_3' \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}(2) \cong \mathbb{C}(2) \tag{23}$$

$$C_4 \cong C_4' \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{R}(2) \cong \mathbb{H}(2) \tag{24}$$

Applying (11), we get:

$$\begin{array}{ll} C_5 \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{H}(2) \cong \mathbb{C}(4) & C_5' \cong (\mathbb{R} \oplus \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{H}(2) \cong \mathbb{H}(2) \oplus \mathbb{H}(2) \\ C_6 \cong \mathbb{H} \otimes_{\mathbb{R}} \mathbb{H}(2) \cong \mathbb{R}(8) & C_6' \cong \mathbb{R}(2) \otimes_{\mathbb{R}} \mathbb{H}(2) \cong \mathbb{H}(4) \\ C_7 \cong (\mathbb{H} \oplus \mathbb{H}) \otimes_{\mathbb{R}} \mathbb{H}(2) \cong \mathbb{R}(8) \oplus \mathbb{R}(8) & C_7' \cong \mathbb{C}(2) \otimes_{\mathbb{R}} \mathbb{H}(2) \cong \mathbb{C}(8) \\ C_8 \cong \mathbb{H}(2) \otimes_{\mathbb{R}} \mathbb{H}(2) \cong \mathbb{R}(16) & C_8' \cong C_8 \cong \mathbb{R}(16) \end{array} \tag{25}$$

We stop at 8, the isomorphisms (12) and (20) together with $C_8 \cong C_8' \cong \mathbb{R}(16)$ imply that, because the C_k are all of the form $F(n)$ or $F(n) \oplus F(n)$, we will just encounter $F(16n)$ or $F(16n) \oplus F(16n)$ for the rest.

2.1.3 Morita equivalence

Before we start studying the modules for the $\mathrm{Cl}_{p,q}$, we need to say a few words about Morita equivalence. Morita equivalence is the appropriate notion of equivalence of algebras if one is interested in studying their representations.

Definition 8. Two algebras are called Morita equivalent if their categories of (left or right) modules are equivalent as categories.

The main result on Morita equivalence we will be using is the following:

Theorem 9. *Let A be an algebra, and let $A(n)$ denote the algebra of $n \times n$ matrices with entries in this algebra. Then A and $A(n)$ are Morita equivalent.*

Another useful result is the following. Observe that $\mathrm{Cl}_{p,q}$ and $\mathrm{Cl}_{p+1,q+1}$ are Morita equivalent, as

$$\mathrm{Cl}_{p+1,q+1} \cong \mathrm{Cl}_{p,q} \otimes_{\mathbb{R}} \mathrm{Cl}_{1,1} \tag{26}$$

and $\mathrm{Cl}_{1,1} \cong \mathbb{R}(2)$, it is generated by the identity, e_1' from (18) and $e_1'e_2'$ from (19). These, together with $e_1'e_1'e_2' = e_2'$ again span $\mathbb{R}(2)$.

Using that the algebra of matrices over an algebra is Morita equivalent to that algebra, we find:

Lemma 10. *$\mathrm{Cl}_{p+1,q+1}$ is Morita equivalent to $\mathrm{Cl}_{p,q}$ for all $p, q \geq 0$.*

Remark 11. A word of caution is appropriate here, naively, one could expect the categories of modules for $C_0 \cong \mathbb{R}$, $C_6 \cong \mathbb{R}(8)$ and $\mathbb{C}_8 \cong \mathbb{R}(16)$ to be equivalent. Note, however, that $\mathbb{R}(8)$ here should not be viewed as simply the 8×8 matrices over $\mathbb{R}$, but rather as a $\mathbb{Z}_2$-graded algebra and we will be interested in studying its $\mathbb{Z}_2$-graded modules.

2.1.4 Bott Periodicity

With this knowledge about Morita equivalences in hand, we can now study the $\mathbb{Z}_2$-graded representations of the Clifford algebras, we will denote the free abelian group generated by the irreducible C_k-modules by $M(C_k)$. Because it is easier to determine non-graded irreducible modules for non-graded algebras, we will reduce to studying $N(C_k^0)$, the free abelian group generated by irreducible ungraded modules for the even part of C_k, viewed as ungraded algebra. This reduction is done via the following two propositions:

Proposition 12. *Assigning to a graded C_k-module $M = M^0 \oplus M^1$ its even part M^0 induces an isomorphism*

$$M(C_k) \cong N(C_k^0). \tag{27}$$

Proof. The map

$$M^0 \mapsto C_k \otimes_{C_k^0} M^0 \tag{28}$$

is an inverse. □

Proposition 13. *We have isomorphisms $C_k \cong C_{k+1}^0$, given by the extension (using 2) of the map $\mathbb{R}^k \to C_{k+1}^0$ given by $e_i \mapsto e_i e_{k+1}$, for $i = 1, \ldots, k$.*

Proof. One checks this map satisfies the condition for the universal property. □

These propositions together give

$$M(C_k) \cong N(C_{k-1}), \tag{29}$$

and as $F(n)$ is simple for F a field, this gives

$$M(C_k) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \text{for } k = 4, 8 \\ \mathbb{Z} & \text{otherwise,} \end{cases} \tag{30}$$

The generators for $N(C_k)$ with $k \neq 3, 7$ are just F^n. The generators for $N(C_3)$ and $N(C_7)$ are the tensor products of the modules $\mathbb{R}^\pm$ for $\mathbb{R} \oplus \mathbb{R}$, defined by acting by ± 1 with the generator e_1', by the modules $\mathbb{H}$ and $\mathbb{R}^8$ respectively. The real dimensions of the even part of the generators for $M(C_k)$ are therefore $1, 2, 4, 4, 8, 8, 8, 8$, respectively.

It is interesting to consider which of these modules for C_k do not come from simply forgetting about the action of one generator in a C_{k+1}-module. This is precisely the cokernel A_k of the map $i^* : M(C_{k+1}) \to M(C_k)$ induced by the inclusion $i : C_k \to C_{k+1}$.

For the cases where $M(C_k) = \mathbb{Z}$, this is easy. Because forgetting a generator of C_{k+1} does not change the real dimension, the dimensions listed above are enough to determine

$$A_k = \begin{cases} \mathbb{Z}_2 & \text{for } k = 1, 2 \\ 0 & \text{for } k = 3, 5, 6, 7. \end{cases} \tag{31}$$

Determining the cokernel of the maps $M(C_{4n+1}) \cong \mathbb{Z} \to M(C_{4n}) \cong \mathbb{Z} \oplus \mathbb{Z}$ takes a bit more effort.

Let us first consider the map $j^* : N(C_2') \to N(C_1')$. The group $N(C_2')$ is generated by $\mathbb{R}^2$ with the usual action of $C_2' = \mathbb{R}(2)$, the group $N(C_1')$ is generated by the $\mathbb{R}^\pm$ described above. The action of the generators of C_2' is given in (18), the map j^* tells us to forget about e_2'. We see that we are left with $\mathbb{R}^+ \oplus \mathbb{R}^-$ and that the cokernel of j^* is $\mathbb{Z}$.

For $N(C_{4n}) \to N(C_{4n-1})$, we employ the isomorphisms (17) and (11). We need to check what these isomorphisms do to j^*. To do this, we note the tensor product of C_2' with C_{2n} is generated by $1 \otimes e_i$ with $i = 1, \ldots, 2n$ and $e_j' \otimes e_1 e_2 \cdots e_{2n}$ with $j = 1, 2$. The map $\hat{j}^* : N(C_2' \otimes C_{4n+2}) \to N(C_1' \otimes C_{4n+2})$ is then just forgetting the action of $e_2' \otimes e_1 e_2 \cdots e_{2n}$. The generators for $N(C_2' \otimes C_{2n})$ and $N(C_1' \otimes C_{2n})$ are just the tensor product modules of the unique irreducible ungraded modules for C_{2n} with $\mathbb{R}^2$ and $\mathbb{R}^\pm$, respectively. On these modules, all generators of $C_2' \otimes C_{2n}$ act block diagonally, except $e_2' \otimes e_1 e_2 \cdots e_{2n}$, so we see that also in these cases the cokernel is $\mathbb{Z}$.

To sum up, we have found that

k	1	2	3	4	5	6	7	8
A_k	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

Table 1: "Periodic table" for Clifford algebras

2.2 Topological K-theory

In this section, we will say a few words about topological K-theory. We will assume the reader is familiar with its definition as given in Atiyah's book [2], and will only recall elements for reference.

2.2.1 Atiyah-Hirzebruch definition

The, perhaps simplest, definition of topological K-theory first defines the relative $K^0(X,Y)$ for a pair of compact spaces $Y \subset X$ to be the Grothendieck group of vector bundles over X that are trivialized on Y. Then one defines the reduced version $\tilde{K}^0(X)$ to be K^0 of X relative to Y. This definition is then extended by reduced suspension Σ to give the negative K-groups $\tilde{K}^{-n}(X) = \tilde{K}^0(\Sigma^n X)$. To extend the definition to include positive groups one invokes Bott periodicity.

2.2.2 Bott Periodicity

It turns out that the functor K^0 from spaces to abelian groups are represented by homotopy classes of maps into BO or BU, depending on whether one is working with real or complex vector bundles. In his classical paper [4], Bott showed that the homotopy groups of the stable classical groups O, U and Sp are periodic. For O, the period is 8, the homotopy groups of O are, starting from π_0, $\mathbb{Z}_2, \mathbb{Z}_2, 0, \mathbb{Z}, 0, 0, 0, \mathbb{Z}$.[1] Note that these are precisely the groups from table 2.1.4. For U, the period is 2, with groups $0, \mathbb{Z}$. This periodicity corresponds to homotopy equivalences $\Omega^8 O \sim O$ and $\Omega^2 U \sim U$ and with that, 8 and 2 periodicity in the real and complex K-groups, respectively. One now just extends periodically to get the positive K-groups.

2.2.3 Bundle sequence model

In comparing Karoubi's model for topological K-theory to Atiyah's model, it will be convenient to use an intermediate model that is shown to be equivalent to the above model in [1]. We will not explain why these are equivalent, but merely state the definition and the alluded to isomorphisms between the K-groups.

The definition will be in terms of sequences of vector bundles. For $Y \subset X$, let $C_1(X,Y)$ denote the set of sequences

$$E = (E_1 \xrightarrow{\sigma} E_0),$$

where σ is an isomorphism defined over Y. Maps of such sequences E and E' are pairs of maps $E_1 \to E_1'$ and $E_0 \to E_0'$ over X that commute with σ over Y. Such a map of sequences is an isomorphism if both maps in the pair are isomorphisms.

Among these sequences there is a special set, the elementary sequences. These are those $E \in C_1(X,Y)$ for which $E_1 = E_0$ and $\sigma = 1$.

Definition 14. Let $L_1(X,Y)$ be the quotient of $C_1(X,Y)$ by the equivalence relation generated by isomorphism up to addition of elementary sequences.

$L_1(X,Y)$ is, a priori, an abelian semi-group under direct sum of sequences. As it turns out, it is in fact a group and:

Proposition 15. *For each $Y \subset X$ compact, we have an isomorphism*

$$L_1(X,Y) \cong K^0(X,Y). \tag{32}$$

The isomorphism is essentially given by sending a class of sequences $[E] = [E_1 \to E_0]$ to the class $[E_0] - [E_1]$.

2.3 Karoubi K-theory

Here, we will give a brief description of part of Karoubi's version of K-theory, as described in his book [9]. We will focus mainly on his incorporation of Clifford algebras into his definition.

[1] Twinkle, twinkle, little star...

2.3.1 Definition

Before giving a definition of Karoubi's model for K-theory we want to consider, it should be pointed out that, in his book, he treats a bunch of quite interesting, categorically phrased, models for K-theory. In this phrasing, relative K-groups are associated to functors between categories of vector bundles over spaces, allowing for a bit more flexibility. He then proceeds to write down a version of this for which the objects carry a $\mathrm{Cl}_{p,q}$-representations[2] and defines $K^{p,q}$ as the relative K-group for the obvious functor between the category of vector bundles with $\mathrm{Cl}_{p,q+1}$-representations and with $\mathrm{Cl}_{p,q}$-representations. This leads naturally to the quotient groups we saw in our treatment of Bott periodicity for Clifford algebras. He then proceeds to show these $K^{p,q}$ can also be described as follows:

Definition 16. Let $K^{p,q}(X,Y)$ be the quotient of

- the abelian semi-group $C^{p,q}(X,Y)$ with generators triples (E,η_1,η_2), where E is a vector bundle over X that carries a $\mathrm{Cl}_{p,q}$-representation and η_1 and η_2 are gradations on E (i.e. bundle automorphisms that square to 1_E and anti-commute with the action off the odd elements of $\mathrm{Cl}_{p,q}$ and commutes with the even elements) that agree on Y,
- by the relations generated by:
  (i) $(E,\eta_1,\eta_2)+(E',\eta_1',\eta_2')=(E\oplus E',\eta_1\oplus\eta_1',\eta_2\oplus\eta_2')$
  (ii) $(E,\eta_1,\eta_2)=0$ if the gradations are homotopic in the space of gradations on E, relative to Y.

This description can, analogous to the description of the Grothendieck group as elements of the form $[E]-[n]$ (here $[n]$ denotes the trivial rank n bundle), be simplified to:

Proposition 17. *Any class in $K^{p,q}(X,Y)$ can be represented by $([r],\eta_{(r)},\eta)$, where $[r]$ is the trivial $\mathrm{Cl}_{p+1,q}$-bundle and $\eta_{(r)}$ is the gradation that comes from viewing the $(p+1)$th generator as a gradation and η is a gradation for the underlying $\mathrm{Cl}_{p,q}$-bundle that agrees with $\eta_{(r)}$ on Y. In this description, $([r],\eta_{(r)},\eta)=([r],\eta_{(r)},\xi)$ if and only if there exists $s\in\mathbb{N}$ such that $\eta\oplus\eta_{(s)}$ and $\xi\oplus\eta_{(s)}$ are homotopic within the space of gradations on $[r+s]$*

Proof. Let (E,θ_1,θ_2) be a triple as in 16. Then E can be viewed as a $\mathrm{Cl}_{p+1,q}$-bundle by viewing θ_1 as the action of the $(p+1)$th generator. This bundle is a projective $\mathrm{Cl}_{p+1,q}\otimes C^0(X)$ bundle, so there is a $\mathrm{Cl}_{p+1,q}$-bundle P, we will call the action of its $(p+1)$th generator ξ, such that

$$(E,\theta_1,\theta_2)+(F,\xi,\xi)=([r],\eta_{(r)},\eta). \tag{33}$$

Now, if two such triples with the same underlying trivial bundle represent the same class, this means they are isomorphic modulo addition of an elementary triple. But such a triple can be assumed to be of the form $([s],\eta_{(s)},\eta_{(s)})$ by the above argument. □

We will often identify the generators of $\mathrm{Cl}_{p+1,q}$ with their action on modules and write $\eta_{(r)}=e_{p+1}$.

In this description, Bott periodicity for K-theory comes pretty much for free, but it turns out the law of conservation of misery has transferred the difficulty one has in proving Bott periodicity in Atiyah's setting to proving the suspension isomorphism.

Before we sketch the proof of the suspension isomorphism, let us have a look at a comparison between Atiyah's model and this one.

2.3.2 Atiyah vs. Karoubi

In his book, Karoubi proves that $K^{p,q}(X,Y)\cong K^{q-p}(X,Y)$, where the latter refers a model for K-theory that is quite clearly equivalent to Atiyah's, for compact X,Y. (This model actually assigns groups to non-compact spaces as well, one should think of this as being analogous to compactly supported ordinary cohomology. Let us give a direct isomorphism between $K^{0,0}(X)=K^{0,0}(X,\emptyset)$ and $L_1(X)=L_1(X,\emptyset)$. The isomorphism for the relative case then follows from the long exact sequences[3] and the five lemma.

[2] Unfortunately, we followed Kitaev's convention on labeling $\mathrm{Cl}_{p,q}$, in Karoubi's book, p and q are switched.

[3] Though we did not discuss this, K-theory has, as a cohomology theory should, long exact sequences for pairs.

Proposition 18. *The group $K^{0,0}(X)$ from definition 16 is isomorphic to the group $L_1(X)$ from definition 14.*

Proof. Elements of the group $K^{0,0}(X)$ are represented by triples (E, η_1, η_2) where E is a vector bundle over X and η_1 and η_2 are $\mathbb{Z}_2$-gradations. Write E_0^i and E_1^i with $i = 1, 2$ for the $+1$ and -1 eigenspaces of η_i, respectively. We define a map on the level of the cochains $C^{0,0}(X)$ and $C_1(X) = C_1(X, \emptyset)$ by[4]

$$\begin{aligned} \theta :& C^{0,0}(X) \to C_1(X) \\ &(E, \eta_1, \eta_2) \mapsto (E_0^1 \to E_0^2), \end{aligned} \tag{34}$$

This map descents to the quotient, isomorphic triples get send to isomorphic triples and an elementary triple is mapped to a sequence isomorphic to an elementary sequence. As an inverse, suppose we are given a sequence $(E_1 \to E_0)$. Then there are bundles F_1 and F_0 of minimal rank such that $E_1 \oplus F_1 \cong E_0 \oplus F_0 \cong [r]$, and we can map the sequence $(E_1 \to E_0)$ to the triple $([r], \mathrm{Id}_{E_1} \oplus (-\mathrm{Id}_{F_1}), \mathrm{Id}_{E_2} \oplus (-\mathrm{Id}_{F_2}))$. This map sends elementary sequences to elementary triples and isomorphic sequences to isomorphic triples so descents to the quotient to give an inverse to the descendant of the map defined above. □

By 10 this implies $K^{k,k}(X, Y) \cong K^0(X, Y)$. Assuming the suspension isomorphism holds for $K^{p,q}$, this shows $K^{p,q}(X, Y) \cong K^{q-p}(X, Y)$.

2.3.3 Suspension isomorphism

We will sketch the proof of the suspension isomorphism for $K^{p,q}$ from Karoubi's book. The proof is by reduction to a theorem of Wood [13] about Banach algebras. The statement of the suspension isomorphism is:

Theorem 19. *View B^1 as $\{e^{i\theta} | \theta \in [0, \pi]\}$ and let $p : X \times B^1 \to X$ be the projection. Define a map*

$$t : K^{p+1,q}(X, Y) \to K^{p,q}(X \times B^1, X \times S^0 \cup Y \times B^1), \tag{35}$$

by setting

$$t(E, \eta_1, \eta_2) = (p^*(E'), \zeta_1, \zeta_2). \tag{36}$$

Here the ζ_i are defined as follows. Let $\epsilon(x)$ denote the gradation making the underlying $\mathrm{Cl}_{p,q}$-bundle of E, denoted by E', into a $\mathrm{Cl}_{p+1,q}$-bundle. Define

$$\zeta_i(x, \theta) = \epsilon(x) \cos(\theta) + \eta_i(x) \sin(\theta), \tag{37}$$

for $(x, \theta) \in X \times B^1$ and $i = 1, 2$.
Then t is an isomorphism.

Note that this map is well-defined, the ζ_i agree on $X \times S^0 \cup Y \times B^1$, as the $\sin\theta$ term vanishes on $X \times S^0$ and the η_i agree on Y, and it clearly sends elementary triples to elementary triples and isomorphic triple to isomorphic triples.

First of all, it turns out we can reduce to the case where $Y = \emptyset$, by a combination of an excision and long exact sequence arguments. The proof now proceeds by progressively moulding elements of $K^{p,q}(X \times B^1, X \times S^0)$ into a form that looks like something in the image of t. As the first step in this, we have proposition 17.

Now, note that given an element of the form $([r], e_{p+1}, \eta) \in K^{p,q}(X \times B^1, X \times S^0)$, with $(\eta|_{X \times S^0} = e_{p+1})$, we can assume that $[r] = p^* E_r$, where E_r is a trivial $\mathrm{Cl}_{p+2,q}$ bundle over X and $\pi : X \times B^1 \to X$ still denotes the projection. The extra generator comes from stabilising the free $\mathrm{Cl}_{p+1,q}$-module of rank r we get from 17 to a the free $\mathrm{Cl}_{p+1,q}$-module. We can use the extra room we made by adding this extra generator to define a gradation over $X \times B^1$, constant along X by

$$\xi_r(\theta) = e_{p+1} \cos\theta + e_{p+2} \sin\theta, \tag{38}$$

[4]Because we are taking sequences relative to the empty set, we do not have to specify a map in our sequences. We stick nonetheless to the notation of definition 14, with the arrow in place.

for $\theta \in B^1$. This at least looks a bit more like the ζ_i from (37). Letting

$$h_r(\theta) = \cos(\theta/2) + e_{p+1}e_{p+2}\sin(\theta/2), \tag{39}$$

we see that $h_r(\theta)$ defines an automorphism bringing our triple $([r], e_{p+1}, \eta)$ into the form $(\pi^* E_r, \xi_r(\theta), h_r(\theta)\eta h_r(\theta)^{-1})$. Now that we have control over the first gradation, we focus on whipping $\eta'(\theta) = h(\theta)\eta h(\theta)^{-1}$ into shape. Because $\eta|_{X\times S^0} = e_{p+1}$ it is not unreasonable to expect that we can write the behaviour of $\eta'(\theta)$ as an automorphism of the trivial bundle acting on e_{p+1}:

Lemma 20. *We can write elements of $K^{p,q}(X\times B^1, X\times S^0)$ as $(\pi^* E_r, \xi_r(\theta), \eta'(\theta))$, where $\eta'(\theta) = f(\theta)e_{p+1}f(\theta)^{-1}$, with $f \in \mathrm{Aut}(\pi^* E_r)$ such that $f(0) = \mathrm{Id}$ and $e_{p+1}f(\pi) = -f(\pi)e_{p+1}$.*

Up to now, we have been a bit sloppy with keeping track of uniqueness of our choices or, equivalently, under what conditions triples in a certain description are isomorphic. As it turns out:

Lemma 21. *An element of $K^{p,q}(X \times B^1, X \times S^0)$ in the form of lemma 20 is zero if and only if, for some $s \in \mathbb{N}$, the automorphism $f \oplus h_s$ is homotopic to h_{r+s} within the subspace of $\mathrm{Aut}(\pi^* E_{r+s})$ of automorphisms satisfying the same conditions as f.*

Note that picking an automorphism as in 20 is the same as picking a path through $\mathrm{Aut}(E_r)$ such that this path begins at Id and ends in the subspace $\mathrm{Aut}^-(E_r)$ of automorphisms that anticommute with e_{p+1}. This choice is only relevant up to homotopy and stabilisation.

Now, E_r is the free rank r module for $\mathrm{Cl}_{p+2,q}\otimes C(X)$ (where $C(X)$ denotes the space of continuous functions on X), viewed as a $\mathrm{Cl}_{p,q} \otimes C(X)$ module. This identifies $\mathrm{Aut}(E_r)$ as $\mathrm{GL}_r(A)$, invertible matrices over the Banach algebra A of endomorphisms of $\mathrm{Cl}_{p+2,q} \otimes C(X)$. We set $\mathrm{GL}_r^-(A) = \mathrm{Aut}_-(E_r)$

Putting this together, we see that $K^{p,q}(X \times B^1, X \times S^0)$ can be viewed as the connected components of the space of paths in the limit $\mathrm{GL}(A)$ over all $\mathrm{GL}_r(A)$ that start at the identity and end in $\mathrm{GL}^-(A)$. Let us call this space $\pi_1(\mathrm{GL}(A), \mathrm{GL}^-(A))$, as it is π_0 of the loop space of the quotient of $\mathrm{GL}(A)$ by $\mathrm{GL}^-(A)$. It turns out this is actually a well-studied invariant of the algebra A. To sum up:

$$K^{p,q}(X \times B^1, X \times S^0) \cong \pi_1(\mathrm{GL}(A), \mathrm{GL}^-(A)). \tag{40}$$

Turning our attention now to the other side of the isomorphism, $K^{p+1,q}(X)$, we see that, on the strength of 17, we can identify $K^{p+1,q}$ with the connected components of the injective limit of gradations of the $\mathrm{Cl}_{p+1,q}$ bundles underlying trivial $\mathrm{Cl}_{p+2,q}$ bundles. But these gradations form a subspace $I(A)$ of $\mathrm{GL}(A)$. All in all, we have

$$K^{p+1,q}(X) \cong \pi_0(I(A)). \tag{41}$$

Tracing through what we did so far, we would find there is a map w from this $\pi_0(I(A))$ to this $\pi_1(\mathrm{GL}(A), \mathrm{GL}^-(A))$ that corresponds to t via all our identifications. We have thus reduced our problem to proving

Theorem 22. *With the notation from above, we have the isomorphism*

$$\pi_0(I(A)) \xrightarrow{w} \pi_1(\mathrm{GL}(A), \mathrm{GL}^-(A)). \tag{42}$$

This theorem was proven by Wood in 1965 for general Banach algebras [13].

3 Hamiltonian formalism for QFT

In this chapter we give a brief introduction to the Hamiltonian formalism, or second quantisation formalism for quantum field theories. This is a formalism is mainly used to talk about many-particle quantum systems.

By a quantum system we will mean a Hilbert space equipped with a Hamiltonian, a (positive-definite) self-adjoint operator. The one parameter subgroup generated by this operator is interpreted as the time evolution of the system. These are very simple examples of quantum systems, but they will suffice for us.

The structure of the exposition is as follows: as a warm-up we first consider a very simple single particle case, that of the harmonic oscillator. Then, we describe what the formalism looks for like a bosonic multi-particle system before we move on to fermions.

3.1 Warm-up: harmonic oscillator

As a warm-up, let us have a look at the basic single particle quantum mechanical system of the harmonic oscillator. The underlying Hilbert space here is (a regularised version of) $L^2(\mathbb{R})$, however, as is often the case, we can ignore this in describing our system. Should the reader be interested, they can view p below as $-i$ times the derivative with respect to the coordinate q on $\mathbb{R}$. A more fruitful way of looking at this system is that we are given an (complex) algebra generated by p and q with the relation that:

$$[p, q] = -i. \tag{43}$$

The Hamiltonian for the quantum mechanical harmonic oscillator,

$$H_{\text{osc}} = \frac{1}{2}(p^2 + \omega^2 q^2), \tag{44}$$

has particularly nice properties: we can define operators

$$\begin{aligned} a &= \sqrt{\frac{\omega}{2}}(q + \frac{i}{\omega}p) \\ a^\dagger &= \sqrt{\frac{\omega}{2}}(q - \frac{i}{\omega}p). \end{aligned} \tag{45}$$

The operator a is called the annihilation operator and $a^\dagger$ the creation operator. satisfying

$$[a, a^\dagger] = \frac{\omega}{2}[q + \frac{i}{\omega}p, q - \frac{i}{\omega}p] = \frac{i}{2}([p, q] - [q, p]) = 1 \tag{46}$$

$$\omega(a^\dagger a + \frac{1}{2}) = \frac{\omega^2}{2}(q^2 + \frac{p^2}{\omega^2} + \frac{i}{\omega}(qp - pq)) + \frac{\omega}{2} = H \tag{47}$$

and

$$[H, a] = \omega[a^\dagger, a]a = -\omega a \tag{48}$$

$$[H, a^\dagger] = \omega a^\dagger[a, a^\dagger] = \omega a^\dagger. \tag{49}$$

Diagonalizing H we obtain a basis $\{|n\rangle\}_{n\in\mathbb{N}}$ of eigenvectors $|n\rangle$ for H with eigenvalues E_n ("energy") for the Hilbert space our operators are acting on[5]. The nice thing is that these vectors together with the creation and annihilation operators form something that looks like a highest weight representation:

$$Ha\,|n\rangle = (-\omega a + aH)\,|n\rangle = (E_n - \omega)a\,|n\rangle\,, \tag{50}$$

$$Ha^\dagger\,|n\rangle = (\omega a^\dagger + a^\dagger H)\,|n\rangle = (E_n + \omega)a^\dagger\,|n\rangle\,. \tag{51}$$

[5] This means we should not take just $L^2(\mathbb{R})$ as our Hilbert space, but a closed separable subspace.

We see that a sends $|n\rangle$ to c_n times some other eigenvector for H presagingly called $|n-1\rangle$, with eigenvalue $E_{n-1} = E_n - \omega$, and $a^\dagger$ sends $|n\rangle$ to b_n times $|n+1\rangle$, with eigenvalue $E_{n+1} = E_n + \omega$, for some $c_n, b_n \in \mathbb{C}$. To determine c_n and b_n, we compute

$$c_n^2 = ||c_n |n-1\rangle ||^2 = \langle n| a^\dagger a |n\rangle = \langle n| \frac{1}{\omega}H - \frac{1}{2} |n\rangle = \frac{1}{\omega}E_n - 1 \tag{52}$$

$$b_n^2 = ||b_n |n+1\rangle ||^2 = \langle n| a a^\dagger |n\rangle = \langle n| \frac{1}{\omega}H + \frac{1}{2} |n\rangle = \frac{1}{\omega}E_n + 1, \tag{53}$$

so, being in a positive mood, we find

$$a^\dagger |n\rangle = \sqrt{\frac{1}{\omega}E_n + 1}\, |n+1\rangle \tag{54}$$

$$a |n\rangle = \sqrt{\frac{1}{\omega}E_n - 1}\, |n-1\rangle . \tag{55}$$

Note that the Hamiltonian is positive definite:

$$E_n = \langle n| H |n\rangle = ||a |n\rangle ||^2 + \frac{1}{2}\omega > 0,$$

where $\langle n|$ denotes the dual of $|n\rangle$ with respect to the inner product on our Hilbert space. So among the eigenvectors, there is a minimal one, $|0\rangle$, with energy $E_0 = \frac{1}{2}\omega$, this vector is referred to as the ground state, vacuum state or heighest weight vector. We see that the eigenvalues E_n for the eigenvectors $|n\rangle$ are given by

$$E_n = \omega(n + \frac{1}{2})$$

and any eigenvector of the Hamiltonian can be obtained by

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n |0\rangle .$$

3.2 Bosons

In this section, we describe the Hamiltonian formalism in the bosonic setting.

3.2.1 Bosonic Fock space

Suppose now that we are given a single particle Hilbert space $\mathcal{H}$ with some single particle Hamiltonian H_0. We can use this Hilbert space to build a state space for arbitrarily many indistinguishable particles in the same system as the direct sum

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} S_\eta \mathcal{H}^{\otimes n},$$

where S_η denotes an (anti)symmetrisation procedure. This is to incorporate the postulate that particles are indistinguishable: any multiple particle state should be invariant under permuting the factors of the tensor product, up to a phase factor $e^{i\eta}$ (this phase factor comes in because, physically, we are only really interested in the projectification of our Hilbert space[6], so any two norm one vectors differing by a phase factor are considered physically equivalent). In principle η can be anything, however, it turns out nature only uses $\eta = 0, \pi$ for elementary particles. The case $\eta = 0$ is that of bosons, $\eta = \pi$ gives rise to fermions. Other values for η do occur in so-called anyons, such particles are said to obey fractional statistics, for these particles $\eta \in \mathbb{Q}\pi$.

[6] It is just computationally cumbersome to work in the projective space.

For bosons, S_0 is just symmetrisation. We pick a vector that spans $\mathcal{H}^{\otimes 0} \cong \mathbb{C}$ and call it $|0\rangle$, the vacuum vector. We will denote homogeneous elements of $\mathcal{F}_n = S_\eta \mathcal{H}^{\otimes n}$ by $|f_1, f_2, \ldots, f_n\rangle$, for bosons, these elements are just the total symmetrisation of $f_1 \otimes f_2 \otimes \cdots \otimes f_n$.

Now that we have a space, we want some operators, specifically ones that will create eigenstates of the single particle Hamiltonian. That turns out to be a bit much to ask for, in some cases this leads to nonsense like creating a "delta function". A better way to say what we want is the following:

Definition 23. A *bosonic creation operator* $\phi^\dagger : \mathcal{H} \otimes \mathcal{F}$ is a distribution on $\mathcal{H}$ with values in the operators on $\mathcal{F}$, with some conditions on the domain of definition which make perfect sense but require going into analysis to state. This operator and its adjoint ϕ satisfy the following

(i) For each $f \in \mathcal{H}$ and $|g_1, g_2, \ldots, g_n\rangle \in \mathcal{F}$, we have $\phi^\dagger[f]\,|g\rangle = |f, g\rangle$.

(ii) For all $f, g \in \mathcal{H}$, we have $[\phi[f], \phi^\dagger[g]] = \langle f, g\rangle_{\mathcal{H}}$ and the creation and annihilation operators for different elements of $\mathcal{H}$ commute, respectively.

Because the creation operators all commute, acting with a linear combination of polynomials in the creation operators on the vacuum automatically gives a symmetrised state and, in fact, we can create all states in $\mathcal{F}$ in this way.

In physics, one usually talks about these operator valued distributions in terms of operators that look like kernels for them. Suppose we are in a setting where $\mathcal{H}$ is a Hilbert space of functions over a manifold M. Then one writes:

$$\phi^\dagger[f] = \int_M dx f(x)\phi^\dagger(x), \tag{56}$$

where, for each $x \in M$, the $\phi^\dagger(x) : \mathcal{F} \to \mathcal{F}$ "creates the function with value 1 over the point x and zero otherwise". This is the nonsense we were referring to above. However, it turns out this language is quite convenient in discussing the gapped free fermion systems later. We will, abusing language, refer to both the operator valued distributions and their kernels as creation and annihilation operators.

Example 24. In the case where $\mathcal{H}$ is a Hilbert space of functions on $\mathbb{R}$ and $H = \Delta + m^2$, the Laplacian plus a constant $m^2 \in \mathbb{R}_+$, the eigenfunctions for the Hamiltonian $|n\rangle$ are just $(x \mapsto e^{ikx})_{k\in\mathbb{R}^*}$ (with eigenvalues $E = -k^2 + m^2$, and the creation operator becomes

$$\phi^\dagger(x) = \int \frac{dk}{\sqrt{2\pi}} a_k^\dagger e^{-ikx}, \tag{57}$$

where $a_k^\dagger$ creates the state $(x \mapsto e^{ikx})$. This is a bit awkward: this corresponds to creating a particle localised at the position x, with wave function given by the Dirac delta function. Physically speaking, this is sort of reasonable: if you want to be able to say where the particle is precisely, the uncertainty principle forces the momentum to be completely undetermined. Mathematically, however, acting with $\phi^\dagger(x)$ on, say, $|0\rangle$ does not represent any vector in the Hilbert space, the integral does not converge.

Like in (56), $\phi^\dagger(x)$ should be viewed as a kernel instead:

$$\begin{aligned}\phi^\dagger[f] &= \int dx f(x)\phi^\dagger(x)\\ &= \int dx \int \frac{dk}{\sqrt{2\pi}} a_k^\dagger e^{-ikx} f(x)\\ &= \int \frac{dk}{\sqrt{2\pi}} \tilde{f}(k) a_k^\dagger,\end{aligned} \tag{58}$$

where $\tilde{f}$ denotes the Fourier transform of f. We can thus think of (57) as being a Fourier inversion formula.

Multiple particle states are created by repeating this process for different wave functions. There is a Hamiltonian $\hat{H}_0$ on the Fock space induced from the single particle Hamiltonian H_0 by

$$\begin{aligned} \hat{H}_0 : \mathcal{F}_1^{\otimes n} &\to \mathcal{F}_1^{\otimes n} \\ v_1 \otimes \cdots \otimes v_n &\mapsto (H_0 v_1) \otimes v_2 \cdots \otimes v_n + \cdots + v_1 \otimes \cdots \otimes H_0 v_n. \end{aligned} \tag{59}$$

The reader should be warned that this construction only gives a small subset of all possible Hamiltonians for many particle systems, see section 3.4.1.

3.3 Fermions

3.3.1 Fermions and anticommutators

As mentioned above, fermions correspond to particles for which the multiple particle states are antisymmetric under permutation of the constituent single particle states. This is often expressed by saying that "fermions obey Fermi-Dirac statistics". Another characterisation of bosons and fermions is that bosons have integer spin and fermions half-integer spin. The equivalence of these two characterisations in a relativistic field theory is expressed by the spin-statistics theorem. In the setting we are in, which is just many-particle quantum mechanics, this theorem does not hold, there are consistent models of spinless fermions.

As a consequence of the anti-symmetrisation, the creation and annihilation operators for fermions will satisfy anti-commutation relations, rather than commutation relations.

3.3.2 Fermionic harmonic oscillator

Let us warm-up by mimicking the harmonic oscillator with anti-commutation relations instead of commutation relations.

The theory will be formulated in terms of fermionic creation and annihilation operators, $b^\dagger$ and b satisfying the anticommutation relations

$$\begin{aligned} \{b, b\} &= 0 \\ \{b^\dagger, b^\dagger\} &= 0 \\ \{b, b^\dagger\} &= 1 \end{aligned} \tag{60}$$

Looking back at our discussion of the harmonic oscillator, we note that if one introduces the so-called number operator $N = a^\dagger a$, the relation between the Hamiltonian and the creation and annihilation operators (47) can be written as:

$$H = \omega(N + \frac{1}{2})$$

and the commutators between the creation and annihilation operators and the Hamiltonian are determined by those for the number operator:

$$[N, a] = -a \tag{61}$$

$$[N, a^\dagger] = a^\dagger. \tag{62}$$

Mimicking this for fermions, setting $N = b^\dagger b$, we get, using (60):

$$\{N, b\} = -b \tag{63}$$

$$\{N, b^\dagger\} = b^\dagger. \tag{64}$$

These relations in turn give the same ladder structure as in the bosonic case, except the ladder is no longer an infinite stairway to heaven but rather a one tier bath stool[7]: the creation operator squares to zero.

Conventionally, the Hamiltonian also changes:

$$H = \omega(N - \frac{1}{2}). \tag{65}$$

[7]Though probably a maharaja bath stool.

3.3.3 Fermionic Fock space

Fermionic Fock space is now built up in the same fashion as that for the bosons, this time we use antisymmetrisation, so the resulting Hilbert space will be a completion of the exterior algebra over the single particle Hilbert space, and the commutation relations are replaced by anticommutation relations.

Remark 25. The structural similarity between the formalisms for bosons and fermions is so strong that many texts on statistical field theory use the notation

$$[a, b]_\pm = ab \pm ba,$$

to write equations for both bosons and fermions at the same time.

3.4 Dynamics

In this section we describe some of the basic features of the dynamics (time evolution) of quantum systems.

3.4.1 Hamiltonians

The Hamiltonian plays a central role in describing the dynamics of a system. Up to now, we have only considered single particle Hamiltonians and many particle Hamiltonians coming from single particle Hamiltonians. In this section we explore what the choices of Hamiltonian are for many particle systems. We will restrict to the case where the Hilbert space $\mathcal{F}$ of our theory is the Fock space for some single particle system $\mathcal{H}$, i.e. comes equipped with creation and annihilation operators.

As a starting point, we take the following definition:

Definition 26. A quantum Hamiltonian on a Fock space $\mathcal{F} = \bigoplus_{n=0}^{\infty} S_\eta \mathcal{H}^{\otimes n}$ is a self-adjoint (not necessarily bounded) operator on $\mathcal{F}$ with spectrum in the positive reals.

From here on, we will use the word Hamiltonian to refer to this definition.

It is convenient to express the Hamiltonian as a power series in the creation and annihilation operators, this is possible as any state in $\mathcal{F}$ can be written as a product of creation operators acting on the vacuum. In this section we will assume we are dealing with an r dimensional single particle Hilbert space, so we have creation operators $a_i^\dagger$ and annihilation operators a_i for $i = 1, \dots, r$. These can be either bosonic or fermionic, though we will only consider the fermionic case when discussing topological insulators. One readily generalises to the infinite dimensional case, our assumption serves to ease exposition.

For notational convenience, we introduce the so-called Majorana operators

$$\begin{aligned} c_{2j-1} &= a_j + a_j^\dagger \\ c_{2j} &= \frac{a_j - a_j^\dagger}{i}, \end{aligned} \qquad \text{for } j = 1, \dots, r. \tag{66}$$

These operators are self-adjoint and satisfy the (anti-)commutation relations

$$[c_j, c_k]_\pm = 2\delta_{jk}. \tag{67}$$

In terms of this basis, the most general Hamiltonian is, naively,

$$H = B + \sum_{j=1}^{2r} iA_j c_j + \sum_{j,k=1}^{2r} iA_{jk} c_j c_k + \sum_{j,k,l=1}^{2r} iA_{jkl} c_j c_k c_l + \dots, \tag{68}$$

where $B \in \mathbb{C}$ and the $A_j, A_{jk}, \dots$ form arrays of complex numbers. The factor i is introduced to match notation with [11]. The choices for these complex numbers are of course restricted by some conditions. First of all, H should be a Hermitian operator in order to ensure its eigenvalues are real or, equally importantly, that iH generates a unitary one parameter family of operators. This means $B, iA_j \in \mathbb{R}$ and

$$\sum_{j,k=1}^{2r} -iA_{jk}^* c_k c_j = \sum_{j,k=1}^{2r} iA_{jk} c_j c_k, \tag{69}$$

so $A = (A_{jk})$ is a skew-Hermitian matrix. Similar conditions can be derived for the other coefficients. Secondly, the linear term can effectively be ignored by a trick that is beyond the scope of this text, so we set $A_j = 0$. Normally, the next step would be to impose a positive energy condition, but it turns out to be convenient for the treatment of topological insulators to work with a Hamiltonian with negative eigenvalues. Assuming there is a lower bound on the eigenvalues, we can use our constant B to shift back to positive values, so the most general Hamiltonian we want to consider is

$$H = \sum_{j,k=1}^{2r} iA_{jk} c_j c_k + \sum_{j,k,l=1}^{2r} iA_{jkl} c_j c_k c_l + \dots, \tag{70}$$

with coefficients such that H is self-adjoint. There are several important possible adjectives for our Hamiltonian:

Definition 27. Let H be as in (70).

- H is a *free*[8] *Hamiltonian* if all terms of order three or higher in (70) vanish.
- H is a *real free Hamiltonian* if $A_{jk} \in \mathbb{R}$ for all j and k.
- H is called a *gapped free Hamiltonian* if there exists a $\Delta > 0$ such that $\lambda^2 > \Delta$ for all eigenvalues λ of iA.
- H is a *bounded, free Hamiltonian* if there is an upper bound on the absolute value of the eigenvalues.

Remark 28. The higher order terms we ask to be zero for a free Hamiltonian would correspond to interactions of particles with themselves, hence the word "free". Note that interaction between different particles is still possible, in through terms like $a_j^\dagger a_k$ with $j \neq k$, but as $A = (A_{jk})$ is diagonalisable this interaction can be ignored by choosing a different basis. Physically, this would correspond to considering composite particles, the particles associated to the initial creation and annihilation operators are usually the elementary particles that make up the quantum system, e.g. electrons in a solid.

When looking at topological insulators, we will not be interested in the specifics of the spectrum of a Hamiltonian, just in its gap structure. It is therefore convenient to introduce the following:

Definition 29. Let H be a gapped, bounded, free Hamiltonian. The *spectral flattening* $\tilde{H}$ of H is the operator $\operatorname{diag}(\operatorname{sign}(E_\lambda)_\lambda)$ in the basis of eigenvectors e_λ with eigenvalues E_λ of H. A gapped, bounded, free Hamiltonian with eigenvalues ± 1 is called *flattened*.

Example 30. Let H come from a single particle fermionic Hamiltonian, so in terms of the creation and annihilation operators, H is given by

$$H = \sum_k E_k b_k^\dagger b_k - \frac{1}{2}. \tag{71}$$

Bringing H into the form above, we find that iA is block diagonal in the c_j-basis, with blocks

$$A_k = \begin{pmatrix} E_k - 1 & iE_k \\ -iE_k & E_k - 1 \end{pmatrix} \tag{72}$$

[8] Note that the terminology is a bit unfortunate at this point, this notion of "free" for many particle systems is different from that for single particle systems, where "free" refers to the absence of a potential.

acting on the span of c_{2k-1} and c_{2k}. These blocks have eigenvalues $i(2E_k - 1)$ and -1. Assuming $E_k > \frac{1}{2}$, flattening this Hamiltonian corresponds to setting $E_k = 1$ for all k. The corresponding matrix A is the block diagonal with blocks

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}. \tag{73}$$

In our treatment of topological insulators we will be interested in Hamiltonians acting on a bundle of Hilbert spaces over some manifold M.

Definition 31. A *fermionic Fock space* over a manifold M is a triple $(\mathcal{F}, \mathcal{H}, \phi^\dagger)$, with $\mathcal{H}$ a Hilbert space with basis indexed by M and some discrete set J, with $\mathcal{F}$ the Hilbert space completion of the exterior algebra over $\mathcal{H}$ and creation operators $\phi_j^\dagger(p) : \mathcal{F} \to \mathcal{F}$ for each $p \in M$ and $j \in J$.

Generalising the above discussion, we find the the most general form for a free Hamiltonian is then:

$$H = i \int_M dx \int_M dy \sum_{j,k=1}^{2r} A_{jk}(x, y) c_j(x) c_k(y), \tag{74}$$

where $A = (A_{jk})$ is now a matrix valued function on $M \times M$, invariant under interchanging its variables. With this generalisation come some extra adjectives:

Definition 32. We will call:

- a *translation invariant Hamiltonian* on $\mathbb{R}^d$ one for which $A(x, y)$ only depends on the difference of its arguments;
- a *local Hamiltonian* one for which A is zero outside the diagonal in $M \times M$.

Remark 33. Many particle Hamiltonians induced from single particle Hamiltonians are necessarily free and local.

For the case $M = \mathbb{R}^d$ ("real space"), it is often advantageous to perform a Fourier transform and view a Hamiltonian on M as a Hamiltonian on M^* ("momentum space") instead. In the case of a free Hamiltonian on M this is (modulo factors of $\sqrt{2\pi}$ and identifying $M^* \cong M$):

$$\begin{aligned} H &= i \int_M dx \int_M dy \int_M dp \int_M dq \int_M dl \int_M dm \sum_{j,k=1}^{2r} A_{jk}(p, q) c_j(l) c_k(m) e^{ix\cdot(p+l)+iy\cdot(q+m)} \\ &= i \int_M dp \int_M dq \int_M dl \int_M dm \sum_{j,k=1}^{2r} A_{jk}(p, q) c_j(l) c_k(m) \delta(p + l) \delta(q + m) \\ &= i \int_M dl \int_M dm \sum_{j,k=1}^{2r} A_{jk}(-l, -m) c_j(l) c_k(m), \end{aligned} \tag{75}$$

with the abuse of notation of writing the same symbol for a function and its Fourier coefficients, distinguishing them by the location of their arguments in the alphabet.

In the translation invariant case, the Fourier transform of $A_{jk}(x, y)$ becomes

$$A_{jk}(x, y) = A_{jk}(x - y) = \int_M dp A_{jk}(p) e^{ip\cdot(x-y)}, \tag{76}$$

and plugging this into the free Hamiltonian above gives

$$H = i \int_M dp A_{jk}(p) c_j(-p) c_k(p). \tag{77}$$

Note that choosing $(x - y)$ over $(y - x)$ was completely arbitrary, this means we have an involution $p \mapsto -p$ on momentum space. In this sense, a translation invariant Hamiltonian on real space Fourier transforms to a local Hamiltonian in momentum space, with this involution. It is these kind of Hamiltonians we will classify later.

3.4.2 Time evolution

The time evolution is given by the flow of the Hamiltonian times i. In a free theory, where we can just diagonalise the Hamiltonian, time evolution of an single particle eigenstate $|n\rangle$ with energy E_n is given by the phase factor $e^{iE_n t}$, this is just the explicit solution to the flow equation

$$\partial_t |n\rangle = H |n\rangle .$$

In the free setting this generalises easily to many particle systems, tensor products of states $|n_1\rangle \otimes \cdots \otimes |n_k\rangle$ evolve according to their total energy $\sum_{j=1}^{k} E_j$ and the rest is just linearity.

3.5 Symmetries

A quantum symmetry is a (projective) (anti-)unitary representation of a (Lie) group on the Fock space for which the generators (anti-)commute with the Hamiltonian. An antiunitary operator is one that can be written as the composition of conjugation on the complex Hilbert space (which is antilinear) and a unitary transformation. Via the expression we have for the time evolution of our quantum system, this notion of symmetry is just the quantum analogue of an Hamiltonian action.

4 Setup of the physics

In this chapter, we set up the necessary framework to formulate the classification results from Kitaev's paper.

4.1 Gapped free-fermion systems

In this section we describe a mathematical framework that captures topological insulators and superconductors. The physical system we are describing is that of electrons (fermions) with a band structure (gapped) neglecting any interactions (free).

4.1.1 Mathematical framework

In order to be able to state our classification problem, we need a definition of what we are trying to classify. Below is a definition that captures the main features. The reader should be aware, however, that there are several things in this definition that need to be made more precise. More details can be found in [5]. We will state our definition in terms of a general manifold. In the physical applications, this manifold will usually be momentum space, i.e. $(\mathbb{R}^d)^*$ or a quotient thereof.

Definition 34. Let d be a positive integer. A *gapped d-dimensional free-fermion system* on a manifold M consists of the following:

- Creation operators (kernels for operator valued distributions) $\hat{\phi}_j^\dagger(k) : \mathcal{F} \to \mathcal{F}$ and annihilation operators $\hat{\phi}_j(k) : \mathcal{F} \to \mathcal{F}$ for all $k \in M$ and $j \in J$. These operators satisfy[9]

$$\{\hat{\phi}_j(k), \hat{\phi}_{j'}^\dagger(k')\} = \delta_{j,j'}\delta(k - k') \tag{78}$$

$$\int_{\mathbb{T}^d} \sum_{j \in J} f^j(k)\hat{\phi}_j^\dagger(k)s = f \wedge s, \tag{79}$$

  for $s \in \mathcal{F}$. Here the $f^j(k)$ are the coefficients of f in the chosen basis for $\mathcal{H}$.

[9] This defines a canonical anti-commutation relation (CAR, its bosonic brother is named after a rock band from the late sixties) representation of $\mathcal{H}$ on $\mathcal{F}$. These are well studied objects in mathematical physics, see for example [6] and references therein.

- An operator $\hat{H} : \mathcal{F} \to \mathcal{F}$, the Hamiltonian, of the form

$$H = \frac{i}{4} \int_M dp \sum_{j,k \in J} A_{jk}(p) \hat{\psi}_j(p) \hat{\psi}_k(p), \tag{80}$$

  where we used the notation

$$\begin{aligned} \hat{\psi}_{2j-1} &= \phi_j + \phi_j^\dagger \\ \hat{\psi}_{2j} &= \frac{\phi_j - \phi_j^\dagger}{i}, \end{aligned} \qquad j = 1, \ldots, n. \tag{81}$$

  We will write $\mathcal{V} = \coprod_{p \in M} \langle \{\hat{\psi}_i(p)\} \rangle$ for the real (or complex) bundle over M formed by taking the point-wise span of these operators. The A from (80) is an anti-symmetric (or anti-selfadjoint) $\mathrm{End}(\mathcal{V})$ valued function on M. This is the most general form for a Hamiltonian that is quadratic in the creation and annihilation operators (free) and local in momentum space.

- For each $p \in M$, the matrix $A(p)$ has eigenvalues ± 1, i.e. the Hamiltonian is gapped.

Remark 35. In what follows, we will mostly focus on the real case of the definition above, the reader should read "(or complex)" everywhere.

Remark 36. One should think about this definition as follows: one is given some single particle quantum mechanical system, parametrised by M (i.e. with Hamiltonian H_0 depending on M). Solving this system corresponds to diagonalising the single particle Hamiltonian $H_0(p)$ for this system, and this in turn gives a basis of eigenstates, parametrised by $p \in M$. The creation and annihilation operators then correspond to creating or annihilating an eigenstate. The multi-particle Hamiltonian for the gapped free fermion system can then be written as a polynomial in these operators.

The single particle Hamiltonian mentioned in the remark above will play an important role in the classification, in the following form:

Definition 37. A *trivial Hamiltonian* for a gapped free fermion system is a Hamiltonian as in (80), with matrix A the block diagonal matrix with

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

on the diagonal, in the basis ψ_i.

One checks this indeed gives the Hamiltonian $\hat{H}_0 = \int_M \sum \phi_j^\dagger(p)\phi_j(p) - \frac{1}{2}$, as one expects to get for a flattened (see below) Hamiltonian in terms of its creation and annihilation operators, compare example 30.

Remark 38. Usually, the Hamiltonian has its eigenvalues in the non-negative reals, in this definition we have shifted its value to make sure the lower energy band and the higher energy band map to opposite sides of zero and performed a (non-unitary) transformation to flatten its spectrum. The latter is justified by the fact that we are not interested in the specific energies of the system, but rather in its algebraic structure. This structure is captured by the (anti-)commutation relations and these are invariant under this flattening.

Remark 39. The condition that the Hamiltonian be local (see definition 32) is necessary to be able to apply the K-theory machinery, as is done in [11]. As mentioned in section 3.4.1, Hamiltonians that are local in momentum space correspond to translationally invariant systems. For readers familiar with this, this might sound like a contradiction with the band insulator story, where the single particle Hamiltonian has a discrete translation invariance. Remember, however, that this translation invariance pertains to the multi particle Hamiltonian, not the single particle Hamiltonian that was the object of study there.

4.1.2 Symmetries

In the above definition we did not account for possible quantum mechanical symmetries of our systems. The flavours we consider are systems with a time-reversal symmetry, a particle number related symmetry or a charge conjugation symmetry. The physical meaning of these symmetries is irrelevant for the classification, so we will immediately turn to an abstraction:

Definition 40. A *symmetry system* for a gapped free fermion system consists of maps $\hat{\Xi}^\mu : \mathcal{V} \to \mathcal{V}$, with μ in some finite index set $\mathcal{I}$ of the form

$$\hat{\Xi}^\mu(\psi_j(p)) = \sum_k \Xi^\mu_{jk} \psi_k(p) \tag{82}$$

with notation as in 34. These operators are subject to the following conditions, for each $p \in M$ the operators Ξ^μ:

- anti-commute with each other and square to $\pm \mathrm{id}$,
- anti-commute with the block diagonal operator from the definition of the trivial Hamiltonian,
- and anti-commute with the map $A(p)$ defining the Hamiltonian, viewed as map $\mathcal{V} \to \mathcal{V}$.

The reader may notice that this comes close to a graded Clifford module structure on the bundle $\mathcal{V}$.

4.1.3 Clifford Algebras

To make the observation at the end of last section precise, consider the following definition:

Definition 41. The *associated Clifford module bundle* for a gapped free fermion system as in 34 with symmetry system as in 40 is the bundle $\mathcal{V}$, equipped with a graded $\mathrm{Cl}_{p,q}$ action, with two different gradings. This action is given by viewing all those Ξ^μ that square to plus or minus the identity as the action of positive or negative generators of $\mathrm{Cl}_{p,q}$, respectively. The gradings are given by the trivial Hamiltonian and the Hamiltonian for the gapped free fermion system.

Thus, associated to any gapped free fermion system we have a Clifford module bundle with two gradings and this bundle captures the relevant physical properties, see the remarks in 4.1.4.

We do have a final issue to resolve, however, before we move on to the classification. K-theory deals with finite dimensional vector bundles and our $\mathcal{V}$ is (depending on the underlying single particle system) possibly infinite dimensional. We should therefore introduce some cut-off, preferably a physically justifiable one. The interesting physics in insulator theory is determined by the part of the system where there is a jump in the spectrum between the valence and conductance bands, the rest of the eigenstates of the Hamiltonian are there, but have either to high or to low an energy to affect the conductance of the material. In our setting, we flattened our Hamiltonian, so it is impossible to say which eigenstates we want to cut off. However, we can restrict our attention to gapped free fermion systems with Hamiltonians that differ from the trivial Hamiltonian by a finite operator and consider the class in K-theory defined by the subbundle associated to this operator. One can think of this undoing a stabilisation in K-theory, justifying this also mathematically.

4.1.4 Conductivity

Although it is beyond the scope of this text to give a thorough explanation, the reader is probably let to wonder what all this has to do with electrical insulators and conductors. Let us at least say a few words about this.

The term insulator comes in, because we are looking at gapped systems. This gap is supposed to correspond to a jump in energy between the valence states (electrons that are stuck at a single spot in the material) and the conducting states (electrons that can move around a solid).

The term topological insulator might be a bit of a misnomer, topological insulators are materials that one would naively expect to be insulators, because they have this energy gap. However, in physics, charge is associated with a parameter in a $U(1)$ gauge transformation. Consider for simplicity a gapped free fermion system as above. Its time evolution is given by the flow of the Hamiltonian, so if we start out with a certain initial state, its time evolution will trace out a section of the bundle $\mathcal{V}$ over M. Suppose now this evolution is periodic, so the state will eventually get back to the same fibre of $\mathcal{V}$. As the Hamiltonian is free, this has to be the same state, up to a phase factor. This phase is partly just e^{iEt}, where E is the energy of the state, but there can also be a part that depends on the particular section that the time evolution traced out. This part of the phase is interpreted as a $U(1)$-monodromy that expresses charge carriage. Without any additional symmetries, this is an integral over the first Chern class of the bundle spanned by the states with energy E over each point in M.

5 Classification of Topological Insulators and Superconductors

Given the remarks from section 4.1.3 and Karoubi's model for K-theory described in 2.3, we are now in a position to classify gapped free fermion systems. Different gapped free fermion systems correspond to different choices of Hamiltonians and with that by different choices of gradations. So via the associated Clifford bundles they correspond, naively, to elements of the K-theory of the space M from definition 34. The only problem is how to deal with non-compact spaces.

5.1 Compact M

The following summarizes our discussion:

Theorem 42. *If M is compact, gapped free fermion systems with symmetries forming a $Cl_{p,q}$ on M are classified by elements of $K^{q-p}(M)$.*

5.2 Non-compact M

To deal with non-compact spaces, we need to say a few words about boundary conditions.

5.2.1 Boundary conditions

We will view boundary conditions as a way to control what happens outside a large compact set. To do this systematically, we will use the following notion:

Definition 43. Let M be a non-compact space, then a set of compact subsets $\{K_i\}_{i\in\mathbb{N}}$, such that

(i) $K_i \subset K_j$ for $i < j$,

(ii) $K_i/\partial K_i$ is homeomorphic to $K_j/\partial K_j$ for all i and j,

(iii) $M = \bigcup_i K_i$,

is called an *exhaustion by compacts* for M.

We can then define:

Definition 44. Let M be a space equiped with an exhaustion by compacts $\{K_i\}$. By a gapped free fermion system $(M, \mathcal{F}, H)$ with *boundary conditions* we mean that, there is an $N \in \mathbb{N}$ such that for each $i > N$, the Hamiltonian satisfies $H(p) = H_0$ for $p \in M - K_i$ and H_0 the trivial Hamiltonian.

Note that a gapped free fermion system with boundary conditions corresponds to an element of any of the relative K-groups $K^{p,q}(K_i, \partial K_i)$ for i large enough. The condition ii on the quotients of the K_i guarantees that all these groups are isomorphic and we can view the element of K-theory as an element of the limit over the inclusions.

5.2.2 Classification for non-compact spaces

For non-compact spaces we have the following:

Theorem 45. *If M is non-compact and equipped with an exhaustion by compacts $\{K_i\}$, gapped free fermion systems with boundary conditions and with symmetries forming a $Cl_{p,q}$ over M are classified by elements of $K^{q-p}(K_i, \partial K_i)$ for some large i.*

5.3 Periodic table

As a cherry on the cake, we can now say where the periodic table of topological insulators and superconductors comes from. Suppose we want to consider translationally invariant (in real space) "gapped free fermion systems" on $\mathbb{R}^d$. Their Hamiltonians will be local in momentum space $\mathbb{R}^d$, so they fit into our definition 34. $\mathbb{R}^d$ has a nice exhaustion by compacts consisting of d dimensional discs of larger and larger radius. Imposing boundary conditions relative to this exhaustion corresponds physically to asking that the only interesting behaviour of the system is at experimentally accessible energy scales. The relative K-groups we are interested in are therefore

$$K^{p,q}(B^d, S^d) \cong K^{q-p}(S^d), \tag{83}$$

and this gives us the wonderful Bott song.

6 Conclusion

6.1 Outlook

6.1.1 Other views on the periodic table

Since it was put on the ArXiV, Kitaev's preprint has been cited around 400 times, of course mostly by physicists, but it has also received some attention from mathematicians. A notable contribution is that of Freed and Moore [7], which emphasises symmetries. This point of view is explored further and explained in [12]. It allows for a nice classification theory in terms of C*-algebras.

6.1.2 Generalisations

There are a few obvious generalisations of the classification problem one can consider.

An essential ingredient in the classification as we presented it here was that our Hamiltonian is local, this allowed us to view the eigenstates as forming a bundle over a parameter space. One can wonder what happens if this locality condition is dropped.

In our discussion, we have only concerned ourselves with free Hamiltonians, i.e. operators that are quadratic in the creation and annihilation operators. If one allows for higher order terms, we can no longer identify the Hamiltonian with a matrix valued function on our space. There are some ideas on how to deal with this case in [7].

6.1.3 State sum models

There are interesting connections between this application of K-theory in condensed matter physics and state sum models. In his paper on the honeycomb model [10], Kitaev highlights some of these connections, inspired by Bellisard's use [3] of Connes' non-commutative geometry to explain the integral quantum hall effect.

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